A family of functions with two different spectra of singularities

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Abstract

Our goal is to study the multifractal properties of functions of a given family which have few non vanishing wavelet coefficients. We compute at each point the pointwise Hölder exponent of these functions and also their local L^p regularity, computing the so-called p-exponent. We prove that in the general case the Hölder and p exponent are different at each point. We also compute the dimension of the sets where the functions have a given pointwise regularity and prove that these functions are multifractal both from the point of view of Hölder and L^p local regularity with different spectra of singularities. Furthermore, we check that multifractal formalism type formulas hold for the functions in that family.

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1 Introduction

Multifractal analysis for signal analysis was developped in the context of fully developped turbulence in order to study the signal of velocity of turbulent fluid, whose regularity is changing from point to point.

Indeed one criterium for estimating the pointwise regularity of a signal at a point x_0 is to compute the pointwise Hölder exponent . Recall its definition.

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Definition 1. *Let $x_0 \in \mathbb{R}^d$ and $\alpha \geq 0$.*
A locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^\alpha(x_0)$ if there exists $C > 0$ and a polynomial P_{x_0} with $deg(P) \leq [\alpha]$ and such that on a neighborhood of x_0,

$$|f(x) - P_{x_0}(x)| \leq C|x - x_0|^\alpha. \tag{1}$$

The pointwise Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$.

Under these assumptions, let $\alpha \in \mathbb{R}$ be fixed and set $E_f(\alpha) = \{x_0 : h_f(x_0) = \alpha\}$. Performing the multifractal analysis of the signal f is computing for every α the Hausdorff dimension $d_f(\alpha)$ of the set $E_f(\alpha)$.

If $E_f(\alpha)$ is non empty for more than two values of α the function is called a multifractal function (if it isn't empty for one single value of α it will be called a monofractal function). The map $\alpha \mapsto d_f(\alpha)$ is called the spectrum of Hölder singularities of the function f.

For example one can see that the classical Weierstrass function

$$f : x \mapsto \sum_{n \in \mathbb{N}} A^n \cos(B^n x) \qquad \text{with } AB > 1 \text{ and } A < 1$$

is a monofractal function, indeed $d_f(\alpha) = -\infty$ for $\alpha \neq -\frac{\ln(A)}{\ln(B)}$ and 1 for $\alpha = -\frac{\ln(A)}{\ln(B)}$. Other examples can be found in numerous works and studies of multifractal functions (see [12] for references).

Other types of pointwise singularities can be studied. Calderon and Zygmund in [3] introduced a local exponent based on L^p norms, the "p exponent".

Definition 2. *Let $p \in [1, \infty]$ and u such that $u \geq -\frac{d}{p}$. Let f be a function in L^p_{loc}. f belongs to $T^p_u(x_0)$ if there exists $R > 0$, P a polynomial such that $deg(P) \leq u$, and $C > 0$ such that*

$$\forall \rho \leq R : \left(\frac{1}{\rho^d} \int_{|x - x_0| \leq \rho} |f(x) - P(x)|^p dx \right)^{\frac{1}{p}} \leq C\rho^u. \tag{2}$$

The p-exponent of f at x_0 is $u^p_f(x_0) = \sup\{u : f \in T^p_u(x_0)\}$

Under these assumptions, let $\alpha \in \mathbb{R}$ be fixed. We denote $E_{f,p}(\alpha) = \{x_0 : u^p_f(x_0) = \alpha\}$ and $d_{f,p}(\alpha)$ the Hausdorff dimension of $E_{f,p}(\alpha)$. The map $\alpha \mapsto d_{f,p}(\alpha)$ is called the spectrum of p-singularities of the function f.

The p exponent was studied in the context of multifractal analysis in [9], [8] for instance, but as far as we know there weren't many contributions where examples of multifractal functions were studied from this point of view. The work

of [6] proves that generically (in the sense of prevalence) in a given functional space like a Sobolev or Besov space, the spectrum of Hölder singularities and the one of p singularities coincide. Indeed the author proves the existence of a prevalent set of functions in a given Sobolev or Besov space (i.e a set whose complement is a Haar-null set) which have the same spectra. Our family of functions don't belong to this prevalent set, especially because the structure of the wavelet coefficients for our family is not at all the same as the one of the functions in this set.
One can easily build functions were the two pointwise exponents are different at least at one given point.

Let f be such that at x_0 $u_p^f(x_0)$ and $h_f(x_0)$ are defined. Then remark that $u_p^f(x_0) \geq h_f(x_0)$. One can check that the p exponent doesn't provide the same local information as the Hölder exponent. For example let $\alpha > 0$ and the function $x \mapsto g(x) = |\,x\,|^\alpha \sum_{j=1}^{\infty} I_{D_j}(x)$ with $D_j = [\,1/2^j - 1/2^{3j},\, 1/2^j\,]$ for $j \geq 0$. The function g satisfies $h_g(0) = \alpha < u_g^p(0) = \alpha + 1/p$ for any $p \geq 1$.

In this work we study a slight modification of the family of functions introduced by S. Jaffard [11], and prove that for each member of this family the spectrum of Hölder singularities and the one of p singularities are different. Actually we will compute at each point of $\mathbb{R}$ the pointwise Hölder exponent, as well as the p-exponent. Whenever the two exponents are different, this is the signature of an oscillating behavior of "chirp" type (see details below) and we will prove that oscillating exponents that detect this kind of behavior are actually non trivial.

Furthermore we will also check that the spectra of singularities satisfy multifractal formalism type formulas. These formulas are generally heuristic formulas whose goal is to compute the spectrum of singularities with the help of global quantities. The first one was introduced by Frish and Parisi in the context of fully developped turbulence [14]. It was then rewritten using wavelet analysis by A. Arnéodo and al. [1]. The domain of validity, counter examples and generic properties of this kind of formulas were the subjects of studies and they are still active fields [15]. We will focus on the multifractal type formulas described in [10] since they fit with the kind of singularities we study. We will prove that these formulas are satisfied by our signals.

The content of the paper is as follows. We present some preliminary notions on wavelets and Hausdorff dimension in Section 2, then the family of functions under study in Section 3, and the main results established thanks to this family of functions in Section 4. Afterwards we provide developments and proofs of the main results separately concerning local regularity, spectra of singularities, and multifractal formalism, in Section 5, 6 and 7 respectively.

2 Definitions and notations

2.1 Wavelet basis

In all the following Λ denotes the set of all dyadic intervals $\lambda = [k2^{-j}, (k+1)2^{-j}]$, $j \in \mathbb{Z}, k \in \mathbb{Z}$ and Λ_j with $j \in \mathbb{Z}$ the subset of dyadic intervals λ of the type $\lambda = [k2^{-j}, (k+1)2^{-j}]$ with $k \in \mathbb{Z}$. We will sometimes write $\lambda = (j, k)$ if no confusion is possible. The notation $[x]$ significates that $[x]$ is the integer part of x, and $\lceil x \rceil$ denotes the smallest integer not less than x.

Recall that a wavelet basis is a set of functions such that ϕ and ψ are functions in $L^2(\mathbb{R})$ and such that they satisfy $\Psi = \{\phi(. - k), k \in \mathbb{Z}\} \bigcup \{2^{j/2}\psi(2^j. - k), j \geq 0; k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. We consider a wavelet basis regular enough, i.e ϕ and ψ are functions in $C^{r+1}(\mathbb{R})$ and with compact support. We will call r the regularity of the basis Ψ. This is always possible (see for example [4] for such constructions).

In order to simplify the notations we will write $\psi_\lambda(x) = 2^{j/2}\psi(2^j x - k)$.

To sum up we have the following equality in $L^2(\mathbb{R})$

$$\forall f \in L^2(\mathbb{R}),\ f(x) = \sum_{k \in \mathbb{Z}} c_k \phi(x - k) + \sum_{j \geq 0} \sum_{\lambda \in \Lambda_j} c_\lambda \psi_\lambda(x) \tag{3}$$

with

$$\begin{aligned} c_k &= \int f(x)\phi(x - k)dx, \forall k \in \mathbb{Z} \\ c_{j,k} = c_\lambda &= \int f(x)\psi_\lambda(x)dx. \end{aligned} \tag{4}$$

2.2 Hausdorff dimension

Definition 3. *Let $A \subset \mathbb{R}^d$; if $\varepsilon > 0$, an ε-covering of A is a countable collection $R = \{A_i\}_{i \in \mathbb{N}}$ such that each diameter $|A_i|$ is less than ε, and $A \subset \bigcup_{i=1}^{\infty} A_i$. If $\delta \in [0, d]$, set*

$$M_\varepsilon^\delta = \inf_R \left(\sum_i |A_i|^\delta \right),$$

where the infimum is taken on all ϵ-coverings R.
For any $\delta \in [0, d]$, the δ-dimensional Hausdorff measure of A is $mes_\delta(A) = \lim_{\varepsilon \to 0} M_\varepsilon^\delta$. There exists $\delta_0 \in [0, d]$ such that

$$\forall \delta < \delta_0, \quad mes_\delta(A) = +\infty \quad \text{and} \quad \forall \delta > \delta_0, \quad mes_\delta(A) = 0;$$

this critical δ_0 is the Hausdorff dimension of A, and will be denoted by $dim(A)$.

Suppose that A is a subset of $\mathbb{R}^d$, and that a numerical quantity $\mathcal{H}(x)$ taking values in $\mathbb{R}^m$ is attached to each point of A. If $\mathcal{H}(x)$ has no regularity, then the level sets of $\mathcal{H}$

$$E_H = \{x : \ \mathcal{H}(x) = H\}$$

may be fractal sets. We consider here the special case where $\mathcal{H}(x)$ is the pointwise Hölder exponent at point x of function f, i.e $h_f(x)$, and for $p \geq 1$ fixed the case where $\mathcal{H}(x)$ is the p-exponent at point x of function f, i.e $u_p^f(x)$. Thus we consider two spectra, which are defined as follows.

Definition 4. *For some function f, we define the Hölder spectrum by:*

$$d_f : h \mapsto dim(E_h), \ \text{with } E_h = \{x : \ h_f(x) = h\},$$

and the p spectrum by:

$$d_f^p : u \mapsto dim(E_u^p), \ \text{with } E_u^p = \{x : \ u_p^f(x) = u\}.$$

3 Functions under study

We define a function f on $[0, 1]$ as a modification of a model by Jaffard [11]. This function has three parameters α, β and γ, with $\alpha \geq 1$ and $\beta \geq 1$ integers and $\gamma > 0$ a non integer. We set

$$f(x) = \sum_{\lambda \in \Lambda(\alpha,\beta)} 2^{-(\gamma+1/2)j} \psi_\lambda(x). \tag{5}$$

In (5), $\Lambda(\alpha, \beta) = \bigcup_{m\geq 1} \Lambda_m^{(\alpha,\beta)}$, where $\Lambda_m^{(\alpha,\beta)}$ is the set of $\lambda = (j, k)$ such that

- $j = \alpha\beta m$, $m > 1$,
- $2^{-j}k = \varepsilon_1\ell_1 + \ldots + \varepsilon_{m-1}\ell_{m-1} + \varepsilon'_m\ell'_m$, where $\varepsilon_i = \pm 1$, $\varepsilon'_i = \pm 1$, $\ell_i = 2^{-i}$ and $\ell'_i = 2^{-\alpha i}$, for each $i > 1$ and $\varepsilon_1 = 1$.

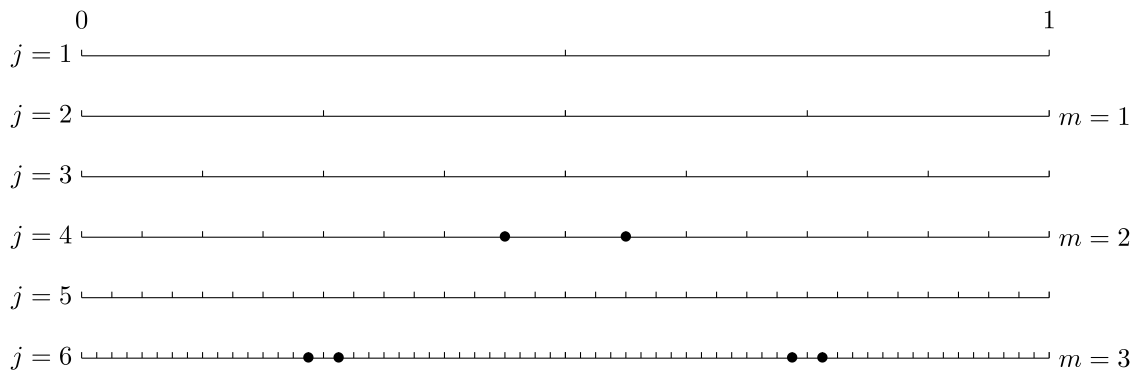
In the sequel, we will denote by c_λ the wavelet coefficients of the function f:

$$c_\lambda = < f, \psi_\lambda > .$$

Let us notice that for $\alpha = 1$, the function is the same as the one of Jaffard [11].

Remark that the definition of f implies that each $m > 1$ creates 2^{m-1} non vanishing coefficients identified by the dyadics $\frac{k}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$, k odd, and the scale $\alpha\beta m$. Its values are all equal to $2^{-(\gamma+1/2)\alpha\beta m}$.

- In the special case where $\alpha = \beta = 1$, this means that at each scale j all the irreducible fractions $\frac{k}{2^j}$, k odd, yield non vanishing wavelet coefficients.



- if we choose $m = 2$ and $j = 4$, then $\frac{k}{2^j} = \varepsilon_1 \ell_1 + \varepsilon_2' \ell_2' = \frac{1}{2} \pm \frac{1}{2^4}$ yields non vanishing coefficients at scale $j = 4$.
- if we choose $m = 3$ and $j = 6$, then $\frac{k}{2^j} = \varepsilon_1 \ell_1 + \varepsilon_2 \ell_2 + \varepsilon_3' \ell_3' = \frac{1}{2} \pm \frac{1}{4} \pm \frac{1}{2^6}$ yields non vanishing coefficients at scale $j = 6$.

Figure 1: Case $\beta = 1$ and $\alpha = 2$. The non vanishing wavelet coefficients • appear on dyadic points $\frac{k}{2^j} = \frac{K}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$ with K odd and $j = \alpha m$.

- In the case where $\beta = 1$ and $\alpha = 2$ each fraction $\frac{k}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$, k odd, is an irreducible fraction at scale $j = \alpha m$.

  This amounts to say that the non vanishing coefficients appear at scale $j = \alpha m$ on irreducible fractions which can be written $\frac{k}{2^j} = \frac{K}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$. Figure 1 provides the repartition of the wavelet coefficients at scale $j = 1, .., 6$ with $\alpha = 2$ and $\beta = 1$.

- In case $\beta > 1$ the fraction of type $\frac{K}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$ is no more irreducible at scale $j = \alpha\beta m$. The coefficient will appear at a finer scale than scale αm. Figure 2) gives an insight of this situation for $\alpha = \beta = 2$.

Following the characterization of $C^\gamma(\mathbb{R})$ (see for example [12]) with the help of wavelet coefficients we have $f \in C^\gamma(\mathbb{R})$ since its wavelet coefficients satisfy: for all $j \geq 0$ and all dyadic interval $\lambda \in \Lambda_j$, $|c_\lambda| \leq 2^{-j(\gamma+1/2)}$.

Remark also that f is compactly supported, thus it is in all L^p spaces for $p \geq 1$. Following [7] (see Chapter 5 Section 5.3), and the fact that f is bounded and in $C^\gamma(\mathbb{R})$, we get that the serie in (5) converges in all L^p for $1 \leq p \leq \infty$.

4 Results

We need to recall some notions which are required to state the results. The first one will be the definition of Hausdorff dimension.

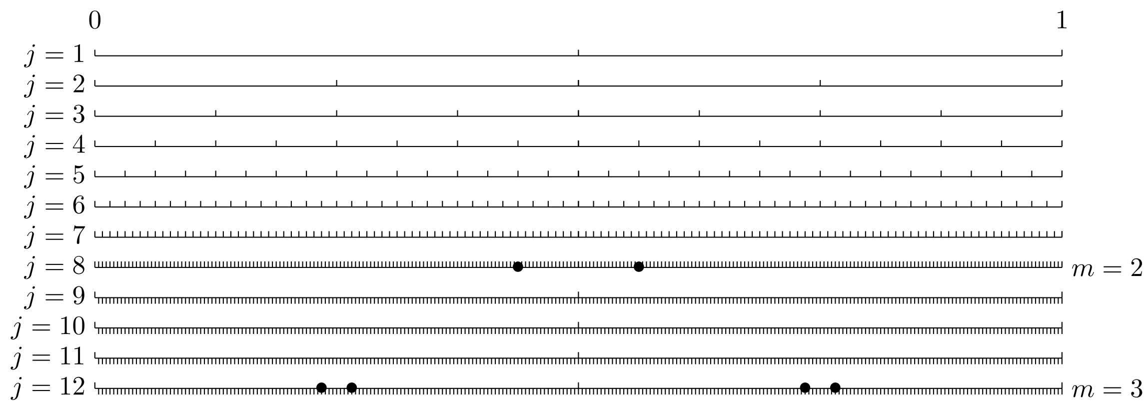


Figure 2: Case $\beta = 2$ and $\alpha = 2$. The non vanishing wavelet coefficients $\bullet$ appear on dyadic points $\frac{k}{2^j} = \frac{K}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$ with K odd and $j = \alpha\beta m$.

4.1 Hölder and p-singularities

One of the key point in multifractal analysis of a function is to understand the underlying structure of the sets E_h and E_u^p. It turns out as we will see in the following that there is a deep connection between these sets and sets of points approximated by special sequences of dyadics. Let us introduce what we mean by these "special sequences".

Let α be an integer larger or equal to 1, and $\mathcal{S}_\alpha$ the set of dyadic points such that $\frac{k'}{2^{j'}} \in \mathcal{S}_\alpha$ if one can find (j,k) such that $\frac{k'}{2^{j'}} = \frac{k}{2^{j-1}} \pm \frac{1}{2^{\alpha j}}$ with $\frac{k}{2^{j-1}}$ an irreducible fraction of order $j-1$. Remark that the set $\mathcal{S}_\alpha$ describes exactly the location of non vanishing wavelet coefficients in f.

We will need the rate of approximation of x_0 by dyadics in this set $\mathcal{S}_\alpha$ given by

$$r_\alpha(x_0) = \limsup_{j' \to \infty} \frac{\log(|K_{j'}(x_0)2^{-j'} - x_0|)}{\log(2^{-j})}, \tag{6}$$

where $K_{j'}(x_0) = \text{argmin}_{k \ / \ k2^{-j'} \in \mathcal{S}_\alpha}(|x_0 - k2^{-j'}|)$ and $(j, k_{j-1}(x_0))$ the integers such that $\frac{K_{j'}(x_0)}{2^{j'}} = \frac{k_{j-1}(x_0)}{2^{j-1}} \pm \frac{1}{2^{\alpha j}}$ with $\frac{k_{j-1}(x_0)}{2^{j-1}}$ an irreducible fraction.

Since we always have $|\frac{K_{j'}(x_0)}{2^{j'}} - x_0| \leq 2^{-(j-1)}$, then $r_\alpha(x_0) \geq 1$.

Remark that dyadic points satisfy exactly $r_\alpha(x_0) = 1$. Indeed if $x_0 = \frac{K_0}{2^{j_0}}$ is a dyadic point then for j' large enough $|x_0 - K_{j'}(x_0)2^{-j'}| = \frac{1}{2^{j-1}} - \frac{1}{2^{\alpha j}}$, which yields $r_\alpha(x_0) = 1$.

Given the definition of $r_\alpha(x_0)$, for every $\delta > 0$ there exists a subsequence m'_n ($m'_n \to \infty$ when $n \to \infty$) and m_n, with $\frac{K_{m'_n}(x_0)}{2^{m'_n}} = \frac{k_{m_n-1}}{2^{m_n-1}} \pm \frac{1}{2^{\alpha m_n}}$ such that,

$$|K_{m'_n}(x_0)2^{-m'_n} - x_0| < 2^{-m_n(r_\alpha(x_0)-\delta)}. \tag{7}$$

Furthermore, still using the definition of $r_\alpha(x_0)$, for every $\varepsilon > 0$, there exists a constant $M > 0$ such that for all $m' \geq M$,there is $m \geq M/\alpha$

$$|K_{m'}(x_0)2^{-m'} - x_0| > 2^{-m(r_\alpha(x_0)+\varepsilon)}. \tag{8}$$

We can now give the results we want to prove in the following.

Theorem 1. *Let α, β and γ, with $\alpha \geq 1$ and $\beta \geq 1$ two integers and $\gamma > 0$ a non integer. Let $p \geq 1$.*

- *Suppose $x_0 \in [0,1]$ and $r_\alpha(x_0) \leq \alpha\beta$ then $h_f(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)}$*

  and $u_f^p(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)} + \left(\frac{\alpha\beta}{r_\alpha(x_0)} - 1\right)\frac{1}{p}$

- *Suppose $x_0 \in [0,1]$ and $r_\alpha(x_0) > \alpha\beta$ then $h_f(x_0) = \alpha\beta\gamma$ and $u_f^p(x_0) = \alpha\beta\gamma + \frac{\alpha\beta-1}{p}$.*

- *$x_0 \notin [0,1]$ then $h_f(x_0) = u_f^p(x_0) = +\infty$*

As a corollary we get

Corollary 2. *The Hölder spectrum of f is the function d_f defined on the interval $[\gamma, \alpha\beta\gamma]$ such that $d_f(h) = \frac{h}{\alpha\beta\gamma}$. The p spectrum is the function $d_{f,p}$ defined on the interval $[\gamma, \alpha\beta\gamma + \frac{\alpha\beta-1}{p}]$ and such that $d_{f,p}(u) = \frac{u+\frac{1}{p}}{\alpha\beta\left(\gamma+\frac{1}{p}\right)}$.*

The proof of Theorem 1 is given in Section 5 and the one of Corollary 2 in Section 6.

The results on the multifractal formalisms can be found in Section 7.

4.2 Oscillation singularities

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5 Local regularity

5.1 Hölder regularity

We first want to study the pointwise Hölder regularity of the function f at each point $x_0 \in \mathbb{R}$. This amounts to compute the pointwise Hölder exponent at $x_0 \in \mathbb{R}$. Recall that we can apply Theorem 1 of [12] which relates the so called wavelet leaders, which depend on the wavelet coefficients of f, with the pointwise Hölder exponent at x_0. We need to start with a definition.

Definition 5. *[12] Two dyadic intervals λ_1 and λ_2 are called adjacent if they are at the same scale and if $dist(\lambda_1, \lambda_2) = 0$ (note that a dyadic interval is adjacent to himself). We denote by $\lambda_j(x_0)$ the dyadic interval of size 2^{-j} containing x_0 and $3\lambda_j(x_0)$ the set of 3 dyadic intervals adjacent to $\lambda_j(x_0)$.*

More precisely if $\lambda = (j, k)$ then we denote $\lambda^l = (j, k-1)$ and $\lambda^r = (j, k+1)$.

Then let

$$d_j(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_\lambda \tag{9}$$

with

$$d_\lambda = \sup_{\lambda' \subset \lambda} |2^{j'/2} c_{\lambda'}| \tag{10}$$

d_λ is called a "wavelet leader".

Theorem 3. *[12] Let $\delta > 0$, $x_0 \in \mathbb{R}$ and f be a function in $L^\infty(\mathbb{R})$. Suppose Ψ is a wavelet basis of regularity $r > [\delta] + 1$.*

- *Suppose f is in $C^\delta(x_0)$. Then there exists $C > 0$ such that*

$$\forall j \geq 0, \; d_j(x_0) \leq C 2^{-\delta j} \tag{11}$$

- *Conversely suppose that (11) holds and furthermore there exists $\varepsilon > 0$ such that $f \in C^\varepsilon(\mathbb{R})$. Then f belongs to $C^{\delta'}(x_0)$ for all $\delta' < \delta$. In particular this means that $h_f(x_0) \geq \delta$.*

- *Suppose $f \in C^\varepsilon(\mathbb{R})$. Then $h_f(x_0) = \liminf_{j \to \infty} \frac{\ln(d_j(x_0))}{\ln(2^{-j})}$*

Since f belongs to $C^\gamma(\mathbb{R})$ we only need to compute, at each point x_0, $d_j(x_0)$ at each scale $j \geq 0$. This is what we is done in Section 5.3.

5.2 L^p pointwise regularity

To study and compute the p exponent at each point x_0 in $\mathbb{R}$, we also compute some quantities related to wavelet coefficients.
Define the so-called p leader

$$D_{\lambda,p} = \left(\sum_{\lambda' \subset \lambda} |c_{\lambda'}|^p 2^{j'(\frac{p}{2}-1)} \right)^{1/p}, \tag{12}$$

We set

$$D_{j,p}(x_0) = \left(\sum_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|^p 2^{j'(\frac{p}{2}-1)} \right)^{1/p}, \tag{13}$$

with the notation $\lambda' = (j', k')$.

It is easy to see that actually $D_{j,p}(x_0) = \left(\sum_{\lambda \in 3\lambda_j(x_0)} D^p_{\lambda,p} \right)^{1/p}$.

Before stating the characterization theorem of [10], we need to recall the characterization of Besov spaces $B^{s,p}_p$ [13].

Theorem 4. *Let $s \in \mathbb{R}$ and $\infty > p > 0$, $q > 0$ and r an integer such that $r > [s] + 1$. Let Ψ be a r regular wavelet basis.*

Suppose f is a tempered distribution with $c_k, k \in \mathbb{Z}$, $(c_{jk})_{j \geq 0; k \in \mathbb{Z}}$ its wavelet coefficients defined by (4).

A tempered distribution f belongs to $B^{s,p}_p$ if (c_k) belongs to l^p and if

$$\sum_{j \geq 0} \sum_k \left| c_{j,k} 2^{(s+1/2-1/p)j} \right|^p < +\infty \tag{14}$$

Remark that a compactly supported function in $C^\varepsilon(\mathbb{R})$ belongs to any Besov space $B^{s,p}_p$ for $s < \varepsilon$.

We have the following theorem of [10] in a slightly modified version in comparison to the original one

Theorem 5. *Let $p \geq 1$ and $u > \frac{-1}{p}$. Let Ψ a r regular wavelet basis with $r \geq [u] + 1$.*

- *Suppose f belongs to $T^p_u(x_0)$ then there exists a constant $C > 0$ such that for all $j \geq 0$*
$$D_{j,p}(x_0) \leq C 2^{-j(u+1/p)} \tag{15}$$
- *Suppose f belongs to $B^{\delta,p}_p$ for some $\delta > 0$. If there exists a constant $C > 0$ such that (15) holds for all $j \geq 0$ then $f \in T^p_{u'}(x_0)$ for all $u' < u$.*
- *Suppose $f \in B^{\delta,p}_p$ for some $\delta > 0$. Then $u^f_p(x_0) = \liminf_{j \to \infty} \frac{\ln(D_{j,p}(x_0))}{\ln(2^{-j})} - \frac{1}{p}$*

Remark that we always have $d_j(x_0) \leq D_{j,p}(x_0)$. This means that whenever the Hölder exponent at a point x_0 of a function f is defined, and if this function satisfies the hypothesis of Theorems 3 and 5 we immediately have

$$h_f(x_0) \leq u^p_f(x_0) \tag{16}$$

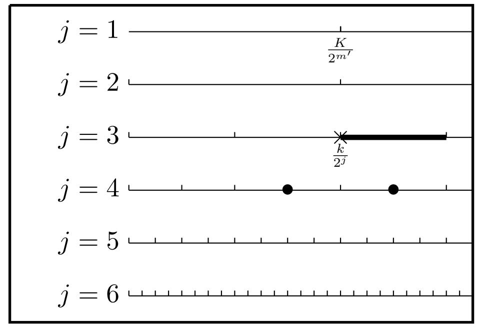
5.3 Study of the pointwise regularity of the function f

5.3.1 Case $\beta = 1$

Wavelet and p leaders Here and after, we will give the methodology for the proof only in the case where $\beta = 1$. This will allows us not to multiplicate the

cases to be considerated. For seek of completeness, the proof when $\beta > 1$ is given in the Apendix.
Let λ be a dyadic interval indexed by (j,k). Let m_1 be an integer such that $\alpha(m_1 - 1) < j \leq \alpha m_1$.

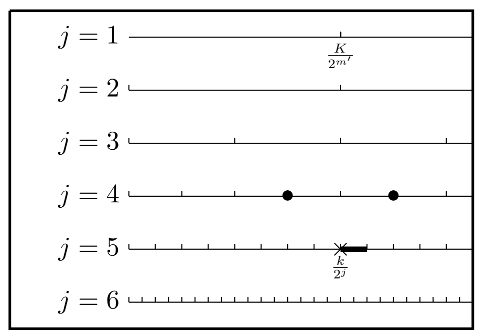
1. Suppose $\frac{k}{2^j} = \frac{K}{2^{m'}}$ is an irreducible fraction with $m' \geq m_1$. It means that the coefficient associated with the irreducible fraction will appear at scale $\alpha(m'+1) \geq j$. So this coefficient will be the first non vanishing coefficient. Then $d_\lambda = \sup_{\lambda' \subset \lambda} |2^{j'/2} c'_\lambda| = 2^{-\alpha\gamma(m'+1)}$.



For what concerns the p-leaders, we know the first non-vanishing coefficient. The following coefficient is located in $\frac{k}{2^j} + \frac{1}{2^j}$ and appears at scale $\alpha(j+1)$ (and it is the same scheme for the followings). We get

$$D^p_{\lambda,p} = 2^{-ap\alpha(m'+1)} + \sum_{\ell=j+1}^{\infty} 2^{\ell-j} 2^{-ap\alpha\ell} = 2^{-ap\alpha(m'+1)} + C2^{-ap\alpha j}.$$

2. Now suppose that $\frac{k}{2^j} = \frac{K}{2^{m'}}$ is an irreducible fraction with $m' < m_1$. That means that the coefficient associated with this irreducible fraction appears at scale $\alpha m' < j$, so before the scale we consider. But in that case, we can notice that $\frac{k}{2^j} + \frac{1}{2^j}$ is irreducible and one of the corresponding coefficient appears at location $\frac{k}{2^j} + \frac{1}{2^j} - \frac{1}{2^{\alpha(j+1)}}$. This yields $d_\lambda = 2^{-\alpha\gamma(j+1)}$.



Again, the p-leader is simply given by

$$D^p_{\lambda,p} = 2^{-ap\alpha(j+1)} + \sum_{\ell=j+2}^{\infty} 2^{\ell-j} 2^{-ap\alpha\ell} = C2^{-ap\alpha j}.$$

Computation of the local regularity of f Let us now prove Theorem 1. Let $x_0 \in \mathbb{R}$ and $p \geq 1$.

1. Let us first remark that for $x_0 \notin [0,1]$, we have for j large enough $d_j(x_0) = 0 = D_{j,p}(x_0)$ since for j large enough all the wavelet coefficients adjacent to $\lambda_j(x_0)$ are vanishing. Thus $h_f(x_0) = u^f_p(x_0) = +\infty$.

2. Let $x_0 \in [0,1]$ and $r_\alpha(x_0)$ defined as in (6). Let $\delta > 0$. As it is explained in Section 2, one can find sequences $m'_n \to +\infty$ and $m_n \to +\infty$ which satisfy (7).

   Let $j_n = [m_n(r_\alpha(x_0) - \delta)]$. Let $m_1^{(n)}$ defined by

   $$\alpha(m_1^{(n)} - 1) \leq j_n < \alpha m_1^{(n)} \quad \text{i.e.} \quad \alpha(m_1^{(n)} - 1) \leq m_n(r_\alpha(x_0) - \delta) < \alpha m_1^{(n)}. \tag{17}$$

   Let k_n such that $\lambda_{j_n}(x_0) = (j_n, k_n) = \lambda_n$.

   Recall that $\lambda_n^l = (j_n, k_n - 1)$, and $\lambda_n^r = (j_n, k_n + 1)$.

   As we already mentioned it, we have $d_{j_n}(x_0) = \sup\{d_{\lambda_n^l}, d_{\lambda_n^r}, d_{\lambda_n}\}$

   On the other hand for $\varepsilon > 0$ one can find M such that for $m' \geq M$ (8) is satisfied.

   Let us consider $3\lambda_j(x_0) = [(k_j - 1)2^{-j}, (k_j + 2)2^{-j}]$. Choose m' the smallest integer such that $\frac{K_{m'}}{2^{m'}} = \frac{k_{m-1}}{2^{m-1}} - \frac{1}{2^{\alpha m}}$ or $\frac{K_{m'}}{2^{m'}} = \frac{k_{m-1}}{2^{m-1}} + \frac{1}{2^{\alpha m}}$ belongs to $3\lambda_j(x_0)$. We have clearly $\alpha m \geq j - 1$.

   Remark also that it is always possible to choose j large enough such that $m' \geq M$.

   Thus
   $$\left| \frac{K_{m'}}{2^{m'}} - x_0 \right| > 2^{-m(r(x_0)+\varepsilon)}$$

   Since $K_{m'} 2^{-m'} \in 3\lambda_j(x_0)$ we have

   $$\begin{aligned} \frac{3}{2^{j+1}} &> 2^{-m(r(x_0)+\varepsilon)} \\ \frac{\ln(3)}{\ln(2)} - 1 + m(r(x_0) + \varepsilon) &> j \end{aligned} \tag{18}$$

   Thus $j \leq m(r_\alpha(x_0) + \varepsilon)$.

   Again define m_1 such that

   $$\alpha(m_1 - 1) \leq j < \alpha m_1 \tag{19}$$

   We consider the following cases

(a) $r_\alpha(x_0) \le \alpha$.

Thus we have immediately $m_n \ge m_1^{(n)}$. Since (7) is satisfied, $\frac{K_{m_n}}{2^{m_n}} \in 3\lambda_{j_n}(x_0)$.

And it is related to Case 2. and yields

$$d_{j_n}(x_0) \ge C2^{-\alpha\gamma m_n}. \tag{20}$$

Thus for any $\delta > 0$

$$\begin{aligned} h_f(x_0) &= \liminf_{j\to\infty} \frac{\log d_{\lambda_j(x_0)}}{\log 2^{-j}} \\ &\le \liminf_{n\to\infty} \frac{\log d_{\lambda_{j_n}(x_0)}}{\log 2^{-j_n}} \\ &\le \frac{\alpha\gamma m_n}{(r_\alpha(x_0)-\delta)m_n} = \frac{\alpha\gamma}{r_\alpha(x_0)-\delta}. \end{aligned} \tag{21}$$

On the other hand we choose $\varepsilon > 0$ small enough so that $r_\alpha(x_0)+\varepsilon < \alpha$ if $r_\alpha(x_0) < \alpha$, and we will have $r_\alpha(x_0)+\varepsilon > \alpha$ if $r_\alpha(x_0) = \alpha$.

i. If $r_\alpha(x_0) < \alpha$ then $\alpha(m_1-1) \le m(r(x_0)+\varepsilon) < \alpha m$ and $m_1 \le m$. This is again related to Case 2. and yields

$$d_j(x_0) \le 2^{-\alpha\gamma m} \tag{22}$$

Thus we get

$$h_f(x_0) \ge \frac{\alpha\gamma}{r_\alpha(x_0)} \tag{23}$$

Thus together with (21) we have $h_f(x_0) = \frac{\alpha\gamma}{r_\alpha(x_0)}$.

Furthermore following the same proof and using the p leader computed in Case 2. we have $u_f^p(x_0) = \frac{\alpha\left(\gamma+\frac{1}{p}\right)}{r_\alpha(x_0)} - \frac{1}{p}$

ii. If $r(x_0) = \alpha$, then we may have $m \le m_1$, with $\frac{K_{m'}}{2^{m'}} = \frac{k}{2^{m-1}} + \frac{1}{2^{\alpha m}}$ inside $3\lambda_j(x_0)$.

This yields in all cases again

$$d_j(x_0) \le 2^{-\alpha\gamma m} \tag{24}$$

Thus we get

$$\begin{aligned} \frac{\ln(d_j(x_0))}{\ln(2^{-j})} &\ge \frac{\alpha\gamma(m+1)}{j} \\ &\ge \frac{\alpha\gamma m}{m(\alpha+\varepsilon)} \\ &\ge \frac{\alpha\gamma}{\alpha+\varepsilon} \end{aligned} \tag{25}$$

Thus together with (21) we have $h_f(x_0) = \frac{\alpha\gamma}{\alpha} = \frac{\alpha\gamma}{r_\alpha(x_0)}$.

The same computation yields $u_f^p(x_0) = \frac{\alpha\left(\gamma+\frac{1}{p}\right)}{r_\alpha(x_0)} - \frac{1}{p}$.

(b) Suppose $r(x_0) > \alpha$. Thus $m_1^{(n)} > m_n$. It is related to Case 1 and yields

$$d_{j_n}(x_0) \geq C2^{-\alpha\gamma j_n} \tag{26}$$

Thus we have the upper-bound

$$\begin{aligned} h_f(x_0) &\leq \liminf_{n\to\infty} \frac{\log d_{\lambda_{j_n}(x_0)}}{\log 2^{-j_n}} \\ &\leq \frac{\alpha\gamma j_n}{j_n} = \alpha\gamma. \end{aligned} \tag{27}$$

For the lower bound we pick up ε small enough and have $r_\alpha(x_0)+\varepsilon > \alpha\beta$. This yields that we may have $m \leq m_1$ and get

$$d_j(x_0) \leq C2^{-\alpha\gamma j} \tag{28}$$

This yields

$$h_f(x_0) \leq \alpha\gamma. \tag{29}$$

Together with (27) we have

$$h_f(x_0) = \alpha\gamma \tag{30}$$

The same computation yields $u_f^p(x_0) = \alpha(\gamma + \frac{1}{p}) - \frac{1}{p}$.

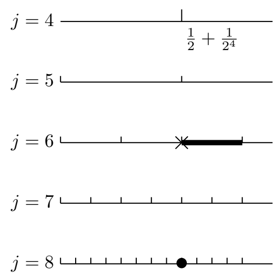
5.3.2 Case $\beta > 1$

Wavelet and p-leaders. The coefficients appear every $\alpha\beta$ scales. The difference in this case is that there is a difference between the scale when the location of the coefficient appears (αm) and the scale when the coefficient really appears ($\alpha\beta(m-1)$). That is why we need to define these two coefficients m_0 and m_1 which verify :

$$\begin{aligned} \alpha\beta(m_0-1) < j \leq \alpha\beta m_0, \\ \alpha(m_1-1) \leq j < \alpha m_1. \end{aligned}$$

That means that when $\beta > 1$, we can have this situation :

0. Suppose that $\frac{k}{2^j} = \frac{K}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$, with $m \geq m_0$. Since we need to have $m\alpha \leq j$, this yields to $m_0 \leq m \leq m_1 - 1$.



Here we consider an example where $\frac{k}{2^6} = \frac{1}{2} + \frac{1}{2^4}$. *So,* $m = 2$, *the location of the coefficient appears at scale* $\alpha m = 4$ *and the coefficient appears at scale* $\alpha\beta m = 8$. *Furthermore, since* $j = 6$, *we have* $m_0 = 2$ *and* $m_1 = 4$ *and then* $m_0 \leq m \leq m_1 - 1$.

Thus, $d_\lambda = 2^{-\alpha\beta\gamma m}$.

For what concerns the $p-$leader, since $m \leq m_1 - 1$, we can have a non vanishing coefficient located at $\lambda' \subset \lambda$ with $\lambda' = (j', k')$ such that $\frac{k'}{2^{j'}} = \frac{k}{2^j} + \frac{1}{2^{\alpha(\alpha m+1)}}$ if $\alpha(\alpha m + 1) \geq j$. Otherwise the first scale $j' > j$ at which a non vanishing coefficient can appear is $\alpha\beta j$. This yields in all the cases

$$\begin{aligned} 2^{-ap\alpha\beta m} + \sum_{\ell=j}^{\infty} 2^{\ell-j} 2^{-ap\alpha\beta\ell} &\leq D^p_{\lambda,p} &&\leq 2 \times 2^{-ap\alpha\beta m} + \sum_{\ell=j}^{\infty} 2^{\ell-j} 2^{-ap\alpha\beta\ell} \\ 2^{-ap\alpha\beta m} &\leq D^p_{\lambda,p} &&\leq 3 2^{-ap\alpha\beta m} \end{aligned} \tag{31}$$

The other cases are very similar to those (1. and 2.) with $\beta = 1$ and we find

1. Suppose that $\frac{k}{2^j} = \frac{K}{2^{m'}}$ is an irreducible fraction with $m' \geq m_1$. Then $d_\lambda = 2^{-\alpha\beta\gamma m'}$ and the p-leader is given by
$$D^p_{\lambda,p} = 2^{ap\alpha\beta m'} + C 2^{-ap\alpha\beta j}.$$

2. Suppose $\frac{k}{2^j} = \frac{K}{2^{m'}}$ (or $\frac{K}{2^{m'}} - \frac{1}{2^j}$) is an irreducible fraction which is not of the type studied in 0., with $m' < m_1$. Thus we will have $d_\lambda = 2^{-\alpha\beta\gamma(j+1)}$, and
$$D^p_{\lambda,p} = C 2^{ap\alpha\beta(j+1)}.$$

Computation of the local regularity of f The only case that really differs from what we have done when $\beta = 1$ is the case $\alpha < r(x_0) \leq \alpha\beta$. This yields $m_0^{(n)} \leq m_n < m_1^{(n)}$.
It is related to Case 0. and yields

$$d_{j_n}(x_0) \geq C 2^{-\alpha\beta\gamma m_n} \tag{32}$$

Thus we get

$$h_f(x_0) \leq \frac{\alpha\beta\gamma}{r_\alpha(x_0)} \tag{33}$$

The same computation yields in the case of the p exponent

$$u_f^p(x_0) \leq \frac{\alpha\beta\left(\gamma + \frac{1}{p}\right)}{r_\alpha(x_0)} - \frac{1}{p} \tag{34}$$

On the other hand we have to consider the following cases.

1. If $\alpha < r_\alpha(x_0) < \alpha\beta$, choosing ε small enough we have $\alpha < r_\alpha(x_0) + \varepsilon$ and $r_\alpha(x_0) + \varepsilon < \alpha\beta$. Using the same notations as in the general setting of this section we get $m_0 \leq m$ but may have $m < m_1$. This is again related to Case 0. and yields

   $$d_j(x_0) \leq C2^{-\alpha\beta\gamma m} \tag{35}$$

   Thus the following upper-bound holds

   $$h_f(x_0) \geq \frac{\alpha\beta\gamma}{r_\alpha(x_0)} \tag{36}$$

   Together with (33) this yields

   $$h_f(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)} \tag{37}$$

   The same computation yields $u_f^p(x_0) = \frac{\alpha\beta(\gamma+\frac{1}{p})}{r_\alpha(x_0)} - \frac{1}{p}$

2. If $r_\alpha(x_0) = \alpha\beta$ remark that the upperbound (33) yields

   $$h_f(x_0) \leq \gamma$$

   Since we already know that $h_f(x_0) \geq \gamma$ because $f \in C^\gamma(\mathbb{R})$ this yields

   $$h_f(x_0) = \frac{\alpha\beta\gamma}{r_\alpha(x_0)} = \gamma \tag{38}$$

For what concerns the p exponent the bound (34) yields

$$u_f^p(x_0) \leq \gamma$$

Since we know already that $u_f^p(x_0) \geq h_f(x_0) = \gamma$ (see (16)) we get

$$u_f^p(x_0) = \frac{\alpha\beta(\gamma + \frac{1}{p})}{r_\alpha(x_0)} - \frac{1}{p} = \gamma \tag{39}$$

This proves Theorem 1.

6 Spectra of singularities

Let $F_r^\alpha = \{x_0 : r_\alpha(x_0) \geq r, r < \infty\}$ and $G_r^\alpha = \{x_0 : r_\alpha(x_0) = r, r < \infty\}$

Remark first that we have

$$\limsup_{j\to+\infty} B(x_j, r_j^r) \subset F_r^\alpha \subset \limsup_{j\to+\infty} B(x_j, r_j^{r-\delta})$$

with $x_j \in S_\alpha, x_j = \frac{k}{2^{j-1}} \pm \frac{1}{2^{\alpha j}}$ and $r_j = 2^{-j}$. Indeed, if $x_0 \in \limsup B(x_n, r_n^r)$, for all $n \in \mathbb{N}$, there exists $j_n \geq n$ such that $x_0 \in B(x_{j_n}, r_{j_n}^r$, so that means that

$$\left| x_0 - \frac{K_{j_n}}{2^{j_n}} \right| \leq 2^{-rj_n}$$
$$\frac{\log |x_0 - \frac{K_{j_n}}{2^{j_n}}|}{\log(2^{j_n})} \geq r,$$

and $r_\alpha(x_0) \geq r$. So we have the first inclusion. For the second one, we use (7) and the conclusion is straightforward.

We want to show that $\dim_H F_r^\alpha = 1/r$. For the upper-bound, it is enough to show that the hausdorff measure of $\limsup\limits_{j\to+\infty} B(x_j, r_j^{r-\delta})$ is finite. Remark that for each scale $j = \alpha\beta m$, there are 2^{m-1} coefficients and the diameters are equal to $|2^{-j}|^{r-\delta}$. Then

$$\sum_{m=2}^{\infty} 2^{m-1} \left(|2^{-j}|^{r-\delta} \right)^{\frac{1}{r-\delta}} = \sum_{m=2}^{\infty} 2^{m(1-\alpha\beta)} < \infty$$

when $\alpha\beta \neq 1$. This yields that the Hausdorff dimension of F_r^α is less or equal than $\frac{1}{r}$.

Several ways are possible in order to get a lower bound of the Hausdorff dimensions. We will make use of recent results by Durand [5] in the version proposed by A. Amou and Y. Bugeaud [2] since this result can be applied directly in our case.

Definition 6. *Let U be a real open interval. Let $(x_i)_{i\geq 1}$ be points in U and let $(r_i)_{i\geq 1}$ be a sequence of positive real numbers such that $\lim\limits_{i\to\infty} r_i = 0$. The family $(x_i, r_i)_{i\geq 1}$ is a homogeneous ubiquitous system in U if the set $\limsup\limits_{i} B(x_i, r_i)$ is of full Lebesgue measure in U.*

Theorem D of [2] proved in [5] yields the following.

Theorem 6. *Let τ be a real number with $\tau \geq 1$. Let the family $(x_i, r_i)_{i\geq 1}$ be a homogeneous ubiquitous system in some open interval U, then the Hausdorff dimension of the set $\limsup B(x_i, r_i^\tau)$ is at least equal to $\frac{1}{\tau}$.*

We have clearly that $\limsup_{j\to+\infty} B(x_j, r_j)$ is of full Lebesgue measure in $(\inf_j x_j - r_j \sup_j x_j + r_j)$ since all points can be approximated by dyadics. Thus the dimension of F_r^α is exactly $\frac{1}{r}$.
Faire le lien avec $G_r(\alpha)$

This yields the spectra and Corollary 2.

7 Multifractal formalism

Let us now check if the function f satisfies a formula of multifractal formalism type.

7.1 Multifractal formalism with Oscillation spaces

Jaffard in [12] (Definition 15) gives a multifractal type formula to compute the Hölder spectrum of singularities, the so called multifractal formalism for Hölder spectrum. This formula, unlike previous formulas which were stated before, is stable under oscillating behaviors and is easy to compute once we have the wavelet leaders d_λ. We will check that it is satisfied in our case.
Recall the definition with the help of wavelet leaders. Indeed we want to compute the following function of q $\omega_f(q) = \sup\{s : \forall j \geq 0, 2^{j(s-1)} \sum_{\lambda\in\Lambda_j} d_\lambda^q < +\infty\}$.
Then the multifractal formalism claims $d(h) = d_O(h)$ with

$$d_O(h) = \inf_q (hq - \omega_f(q) + 1)$$

Let us check if this formula is true for our function f.
As usual define $m_0 = \lceil \frac{j}{\alpha\beta} \rceil$ and $m_1 = \lfloor \frac{j}{\alpha} \rfloor$. We have 2^{j+1} dyadics intervals at scale j inside $[0,1]$ with 2^m irreducible fractions of type $\frac{k}{2^{m-1}} \pm \frac{1}{2^{\alpha m}}$ for $m_0 \leq m \leq m_1$ with a general count of $2^{\alpha m+1}$ of irreducible fractions at scale αm.
Let

$$2^{j(s-1)} \left(\sum_{m=0}^{m_1-1} 2^{m+1} 2^{-q\alpha\gamma\beta(j+1)} + \sum_{m=m_1}^{j} 2^{m+1} 2^{-\alpha\beta\gamma q m} + \sum_{m=m_0}^{m_1-1} 2^m 2^{-\alpha\beta\gamma q m} \right) = \Omega_f(j,q) \quad (40)$$

Some dyadic intervals are counted twice in $\Omega_f(j,q)$. But we can say that we have anyway

$$\frac{1}{2}\Omega_f(j,q) \leq 2^{j(s-1)} \sum_{\lambda\in\lambda_j} d_\lambda^q \leq \Omega_f(j,q) \quad (41)$$

we have the following cases

1. Suppose $1-\alpha\gamma\beta q < 0$, thus $\frac{1}{\alpha\gamma\beta} < q$ then we have

   $\omega_f(q) = -\frac{1}{\alpha\beta} + q\gamma + 1$

2. Suppose $1-\alpha\gamma\beta q \geq 0$, thus $\frac{1}{\alpha\gamma\beta} \geq q$ then we have,

   $\omega_f(q) = q\alpha\beta\gamma$

Let h be fixed and $f(q) = hq - \omega_f(q) + 1$. We have

$$f(q) = \begin{cases} hq - \gamma\beta q + \frac{1}{\alpha\beta} & \text{if } \frac{1}{\gamma\beta\alpha} < q \\ hq - \alpha\beta\gamma q + 1 & \text{otherwise} \end{cases} \tag{42}$$

- Suppose $h > \alpha\gamma\beta$ or $h < \gamma$. Thus f is unbounded from below and $d_O(h) = -\infty$.
- Suppose $\gamma \leq h \leq \alpha\gamma\beta$. The minimum of f is at $p_0 = \frac{1}{\alpha\gamma\beta}$ and we have $f(p_0) = \frac{h}{\alpha\gamma\beta}$

Since $d_f(h) = d_O(h)$ for all h the multifractal formalism with oscillation spaces is satisfied.

7.2 Multifractal formalism with p-Oscillation spaces

The same kind of formula as in the Hölder case exists in order to compute the p spectrum. The claim is the following (see [10] for details): compute $\omega_f(p,q) = \sup\{s : \forall j \geq 0, 2^{j(s-1)} \sum_{\lambda\in\lambda_j} D^q_{\lambda,p} < +\infty\}$.

Then

$$d_p(h) = \inf_q (hq - \omega_f(p,q) + 1)$$

should give $d_p(h) = d_{f,p}(h)$.

We can check that this is actually true for our function f. Indeed remark that it is enough to replace γ by a in the previous computation of Section 7.1 to compute exactly the formula for the p spectrum and get it exactly. The multifractal formalism for the p exponent is satisfied.

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